Final Exam Quantum Physics 1 – 2025/2026

Wednesday, October 29, 2025, 18:15 – 20:15

Read these instructions carefully. If you do not follow them your exam might be (partially) voided.

- This exam consists of 3 questions in 2 pages and a formula sheet at the end.
- The points for each question are indicated on the left side of the page.
- You have 2 hours to complete this exam.
- **Write your name and student number on *all* answer sheets that you turn in**.
- Start answering each exercise on a new page. It is ok to use front and back.
- Clearly write the total number of answer sheets that you turn in on the first page.
- Telephones, smart devices, and other electronic devices are **NOT** allowed.
- **This is a closed book exam**. Consulting reading material is **not** allowed.

33 pts **Question 1 – Harmonic Oscillator**

For this question consider the one-dimensional quantum harmonic oscillator for a frequency ω and a particle with a mass m. The Hamiltonian is given by: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$.

The raising and lowering operators are defined as: $a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x \mp ip)$. When applied to the energy eigenstates ψ_n, we have: $a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$ and $a_-\psi_n = \sqrt{n}\,\psi_{n-1}$.

The Hamiltonian can be conveniently expressed as a function of the ladder operators as:

$H = \hbar\omega\left(a_\pm a_\mp \pm \frac{1}{2}\right)$.

8 pts **a)** Show that $[a_-, a_+] = 1$ by making use of the canonical commutator.

5 pts **b)** The operator $N = a_+a_-$ is also known as the "number operator". Show that ψ_n, i.e. the energy eigenstates, are also eigenstates of the number operator. Use the properties of the ladder operators given above to help you answer this question.

Then, express the Hamiltonian in terms of N and show that the eigen-energies are given by: $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$.

12 pts **c)** Suppose that we place a particle in the state:

$$\psi = \frac{1}{\sqrt{2}}(\psi_0 + i\,\psi_1),$$

where ψ_n (n=0, 1, 2, ...) are the energy eigenstates.

Find the expectation value for position and momentum.

Hint: If you make full use of the ladder operators you won't need to calculate much.

8 pts **d)** We now place two electrons in the harmonic oscillator and let them relax to the ground state. Write down the complete quantum state of this two-particle system (including spin).
How would your answer change if we place one particle in the ground state and the other in the first excited state?

33 pts **Question 2 – Spin 1/2**

An electron is in the spin state $|\chi\rangle = A(\,2i\,|\uparrow\rangle + 3|\downarrow\rangle\,)$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of the operator $\hat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

3 pts **a)** Determine the constant A.

5 pts **b)** What are the possible outcomes of a measurement of the spin in the z direction and with which probabilities? If you haven't been able to determine the constant A in the previous question you can leave your answer as a function of A.

10 pts **c)** Is the state $|\chi\rangle$ an eigenstate of $\hat{S}_z$? What about $\hat{S}_x$? Justify your answers by explicitly applying the operators to the state $|\chi\rangle$. Use the relations you found to calculate the expectation values for the operators $\hat{S}_x$ and $\hat{S}_z$.
Tip: You can use the matrix form of the operators.

8 pts **d)** Show that the eigenstates of $\hat{S}_x$ are given by $|\uparrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|\downarrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$.

7 pts **e)** What are the possible values of a measurement of $\hat{S}_x$ in the state $|\chi\rangle$ stated above? With which probabilities? With no further calculations, give the expectation value for $\hat{S}_x$. You can use your answer here to cross-check your answer in **(c)**.

34 pts **Question 3 – Trapped and free particles**

For this question, consider a particle in one-dimension. We first place a particle in a delta function potential well: $V(x) = -\alpha\,\delta(x)$, with α being a positive constant.

9 pts **a)** Show that there is only one bound state with wavefunction $\psi(x) = A\,e^{-\kappa|x|}$, where A and κ are constants. Also determine the normalization constant A.

8 pts **b)** What is the energy of this bound state? You can leave the answer as a function of A if needed. You will need to integrate the Schrödinger equation around the potential well, from $-\varepsilon$ to ε and take the limit $\varepsilon \to 0$ to find E.

Now we will consider the free particle, i.e. the potential now is $V(x) = 0$ everywhere.

9 pts **c)** In the course we have learned that we can easily change the basis of our representation through the use of projection operators.
Knowing that the wavefunction of a particle in a quantum state $|\alpha(t)\rangle$ at time t in position-representation is given by $\Psi(x,t) = \langle x|\alpha(t)\rangle$, use the projection operator for the eigenstates of $\hat{x}$ in the bra-ket notation form – i.e. $|x\rangle$ – to show that the wave function in momentum space representation is given by $\Phi(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,t)e^{-ikx}\,dx$.
Remember that the momentum eigenstates in x-representation are given by $\langle x|k\rangle = \frac{1}{\sqrt{2\pi}}e^{ikx}$.

8 pts **d)** Find $\Psi(x,t)$ at later times if the wavefunction of the free particle at $t = 0$ is given by the same as in question **(a)** – i.e. $\psi(x, t=0) = A\,e^{-\kappa|x|}$.

You only need to express the integral which needs to be solved to find $\Psi(x,t)$ but <u>do explicitly find the expression for ϕ</u> as a function of κ, k and A. Note that for $t = 0$ one does not need to know ω to find the expression for ϕ.

Useful formulas:

Schrödinger equation	$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$
Time-independent Schrödinger equation	$H\psi = E\psi \qquad \Psi = \psi e^{-iEt/\hbar}$
Hamiltonian operator	$H = -\frac{\hbar^2}{2m}\nabla^2 + V$
Momentum operator	$p = -i\hbar\,\nabla$
De Broglie wavelength	$\lambda = h/p$
Time-dependence of expectation value	$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[H,Q]\rangle + \langle\frac{\partial Q}{\partial t}\rangle$
Generalized uncertainty principle	$\sigma_A\sigma_B \geq \left\lvert\frac{1}{2i}\langle[A,B]\rangle\right\rvert$
Heisenberg Uncertainty principle	$\sigma_x\sigma_p \geq \hbar/2$
Canonical commutator	$[x,p] = i\hbar$
Angular momentum	$\vec{L} = \vec{r}\times\vec{p}$
	$[L_x, L_y] = i\hbar L_z\ ;\ [L_y, L_z] = i\hbar L_x\ ;\ [L_z, L_x] = i\hbar L_y$
In spherical coordinates:	$L_z = -i\hbar\ {\partial}/{\partial\phi}$
	$L_x = -i\hbar\left(-\sin\phi\ {\partial}/{\partial\theta} - \cos\phi\cot\theta\ {\partial}/{\partial\phi}\right)$
	$L_y = -i\hbar\left(+\cos\phi\ {\partial}/{\partial\theta} - \sin\phi\cot\theta\ {\partial}/{\partial\phi}\right)$
Ladder operators:	$L_\pm\lvert l,m_l\rangle = \hbar\sqrt{l(l+1) - m_l(m_l\pm 1)}\,\lvert l,m_l\pm 1\rangle$
	$L_\pm \equiv L_x \pm iL_y$
Pauli matrices	$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ ;\ \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\ ;\ \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
Position-space and momentum-space	$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k,t=0)e^{i(kx-\omega t)}dk$
	$\phi(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,t=0)e^{-i(kx-\omega t)}dx$
Trigonometric relations	$\sin(a\pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$
	$\cos(a\pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$